Dynamic Analysis of Structural Systems Using Component Modes

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A method is developed for analyzing complex structural systems that can be divided into interconnected components. Displacements of the separate components are expressed in generalized coordinates that are defined by displacement modes. These are generated in three categories: rigid-body, "constraint," and "normal" modes. Rigid-body modes are convenient where displacements are defined in inertial space for dynamic analysis. "Constraint" modes are included to treat redundancies in the interconnection system. "Normal" modes define displacements relative to the connections. Generalized mass, stiffness, and damping matrices are determined for each component, as are generalized forces. The requirement of system continuity gives rise to equations of displacement compatibility at the connections. These serve as equations of constraint among the component coordinates and are used to construct a transformation relating component coordinates to system coordinates. This transformation is used to derive system properties and forces from component properties and forces. System equations of motion are formulated and solved to determine system response. Component responses are found using the transformation. Connection forces are computed from the component equations. Each component can then be isolated and treated separately.

Introduction

IN some of the first works on matrix methods of structural analysis, authors such as Argyris[1] and Turner[2] dealt with complex structures by subdividing them into components that were treated separately to produce results that were then synthesized to obtain elastic properties of the complete structure. Indeed, the process of synthesis related to properties of the separate elements of a structural system is basic to the displacement and force methods of analysis as treated by matrix methods. The generalization of this process to include, as an intermediate step, the synthesis of properties of major components of a structural system would seem to be a trivial step to take. However, it involves special problems that must be resolved and that have caused special emphasis to be given to the subject. Early works by Argyris that relate to this subject are collected and extended in Refs. 1 and 4. Here the primary emphasis is directed toward force methods, and the problem of dealing with indeterminate connection systems among the components is handled by considering the equilibrium of interaction redundant force systems. However, displacement methods are considered also, and in these methods it is suggested that the interconnection problem may be resolved by equating matching boundary displacements. Studies by Turner, Martin, and Weikel[3] consider the analysis of complex structures by stiffness or displacement methods in which major components are treated separately as free bodies. In a more recent paper,[5] Przemieniecki developed a method of substructures using displacement unknowns in the analysis, thus placing it in the category of displacement methods. In this method each substructure is analyzed with all displacements on the common boundaries between adjacent components completely constrained. These boundaries are subsequently relaxed, and their displacements are determined by the condition of equilibrium of the boundary forces.

This paper has its genesis in an earlier study[6] in which natural modes of vibration of structural systems were found by using displacement mode functions related to the components. In the examples treated, the systems were frame structures and the components were beams. A subsequent study[7] was made for the purpose of generalizing the method and extending it to include other types of structures. Of special interest were space vehicle structures. Also, it was considered desirable to standardize the analysis procedures insofar as possible so as to facilitate programing for solution by digital computers.

Subsequent to the completion of the report in Ref. 7, the author became aware of still another method of analysis which is, in some respects, quite similar to that reported in this paper. Developed by Gladwell,[8] this method involves the imposition of a sequence of constraints on the system so that, for each constrained system or branch, only a few adjoining components vibrate in modes that are called branch modes. These branch modes, together with appropriate rigid-body modes, are employed in a Rayleigh-Ritz analysis of the complete system.

As it has evolved, the method treated here presents an approach in which displacements of the components are defined in terms of generalized coordinates that are related to specified sets of normalized displacement functions or modes. Therefore, this approach is properly classed as a displacement method. Among other methods presented in the literature to date, it is somewhat unique in the use of these displacement modes and, in particular, in the manner in which they are classified and used. They are considered in three categories as follows. First are rigid-body displacements in which the component is displaced without deformation. If no fixed external constraints are imposed on the component, it will have six degrees of freedom as a rigid body; hence there may be as many as six rigid-body modes for the component. This number will be reduced if external constraints are present. In the second category are modes that will exist only if the system of constraints on the component is indeterminate. These modes are defined by producing a unit displacement on each redundant constraint in turn, with all

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other constraints fixed. For brevity, these modes are called "constraint" modes, and they are equal in number to the number of redundant constraints. The third category includes modes that define displacements relative to the constraint system. They may be, for example, the normal modes of vibration of the component with all constraints fixed. For this reason they are called "fixed-constraint normal modes" or, simply, "normal modes" for brevity. All of these displacement modes define, for each component, a set of generalized coordinates in which Lagrange equations are formulated.

As related to each separate component these equations constitute equations of motion expressed in terms of a set of independent, generalized coordinates. However, when the continuity conditions at the interconnections are imposed, a set of constraint equations results which expresses kinematic dependencies among the coordinates related to the various components. These equations of constraint are used to determine a set of system generalized coordinates equal in number to the total number of component coordinates minus the number of equations of constraint. The relationship between the sets of component generalized coordinates and the set of system generalized coordinates is expressed by a transformation matrix $[\beta]$. Mass, elastic, and damping properties of the system are obtained from the corresponding properties for the separate components by use of this transformation. Also, component forces are transformed into system forces by the same transformation. Thus, a set of equations of motion related to the system is formulated. This set of equations is solved to determine the system response. Through the forementioned transformation the component responses are then derived. Following this, each component may be analyzed separately to determine all the constraint forces imposed upon it. Also, dynamic stresses, strains, deflections, or other responses may be determined as desired.

Equations of Motion

Equations of motion for an undamped, linear, structural system, written in terms of generalized coordinates, have the following matrix form:

$$[M]\{\ddot{q}\} + [K]\{q\} = \{Q(t)\} \tag{1}$$

where

$\{q\}$ = column matrix of generalized displacements
$\{\ddot{q}\}$ = column matrix of generalized accelerations
$[M]$ = square matrix of system generalized masses
$[K]$ = square matrix of system generalized stiffnesses
$\{Q(t)\}$ = column matrix of time-dependent generalized forces

For a damped system this equation will be modified by the addition of a damping term. For linear viscous damping the matrix equation of motion is written as

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{Q(t)\} \tag{2}$$

where

$\{\dot{q}\}$ = column matrix of generalized velocities
$[C]$ = square matrix of system generalized damping coefficients

For structural damping it is generally assumed that the existence of damping does not cause coupling of the undamped natural modes of vibration. Therefore, Eq. (1) is used to determine natural modes, and structural damping factors appropriate to each of the modes are subsequently introduced into the normal-mode response equations.

Equations (1) and (2) apply to the complete structural system. Equations of similar form apply also to each separate component. For example, the following equation applies to the sth component in the case of viscous damping:

$$[m]_s\{\ddot{p}\}_s + [c]_s\{\dot{p}\}_s + [k]_s\{p\}_s = \{P(t)\}_s \tag{3}$$

where

$\{p\}_s$, $\{\dot{p}\}_s$, $\{\ddot{p}\}_s$ = column matrices of component generalized displacements, velocities, and accelerations, respectively
$[m]_s$, $[c]_s$, $[k]_s$ = square matrices of component generalized masses, damping coefficients, and stiffnesses, respectively
$\{P(t)\}_s$ = a column matrix of generalized forces applied to the sth component. These include forces transmitted to it through the constraints as well as those applied from sources external to the system

Equations similar to Eq. (3) are written for all components of the system and all of these equations are combined in the following single matrix form:

$$[m]\{\ddot{p}\} + [c]\{\dot{p}\} + [k]\{p\} = \{P(t)\} \tag{4}$$

In writing this equation it is desirable to group the coordinates relating to each component; thus,

$$\{p\} = \begin{Bmatrix} \{p\}_1 \\ \{p\}_2 \\ \cdot \\ \cdot \\ \cdot \\ \{p\}_r \\ \{p\}_s \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \qquad \{P(t)\} = \begin{Bmatrix} \{P(t)\}_1 \\ \{P(t)\}_2 \\ \cdot \\ \cdot \\ \cdot \\ \{P(t)\}_r \\ \{P(t)\}_s \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \tag{5}$$

Components r and s are considered as typical components. With the coordinates so grouped, the mass matrix in Eq. (4) has the following form:

$$[m] = \begin{bmatrix} [m]_1 & & & & & & & \\ & [m]_2 & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & [m]_r & & \\ & & & & & & [m]_s & \\ & & & & & & & \cdot \\ & & & & & & & & \cdot \end{bmatrix} \tag{6}$$

All elements in this matrix not otherwise designated are zeros. The damping and stiffness matrices have similar forms. Physically interpreted, Eq. (4) can be considered as a set of equations of motion for the group of components not connected together. The physical process of connecting them gives rise to equations of constraint among the elements of the matrix $\{p\}$. If there are m components in the vector $\{p\}$ and k equations of constraint relating them, then there will exist a subset of these components containing $n = m - k$ displacements that are independent. This subset may be identified directly with the set of system generalized displacements $\{q\}$, or it may be related to that set by a linear transformation. In either case a transformation can be derived that relates the vector $\{p\}$ to vector $\{q\}$:

$$\{p\} = [\beta]\{q\} \tag{7}$$

The transformation matrix $[\beta]$ is of order $m \times n$ where $m > n$. Since the q's are independent, the system has n degrees of freedom.

Construction of the matrix $[\beta]$ requires knowledge of the constraints imposed on all components by the system of connections. Suppose that at a certain connection between two components r and s a constraint exists which requires that the translations of a common point on the two components be equal:

$$\bar{u}^r = \bar{u}^s$$

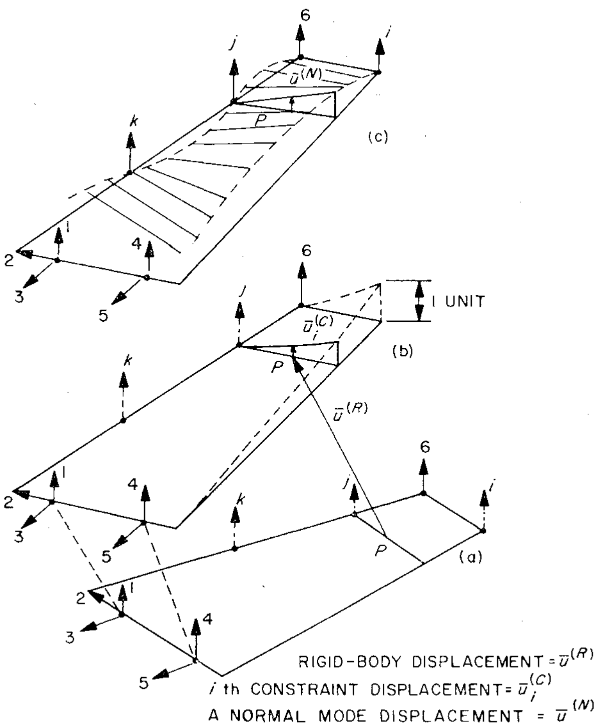

Fig. 1 "Plate-like" structural component.

Another constraint may require that rotations be equal:

$$\bar{\psi}^r = \bar{\psi}^s$$

Representation of finite rotations as vectors is acceptable only for small rotations. In this sense, and in other respects as well, the analysis is limited to small deflections. The displacement vectors in the foregoing equations may be expressed in the local coordinates established for the two components. A transformation of coordinates will permit writing these constraint equations in terms of displacement components on a common axis set. These displacement components may be expressed in terms of the component generalized coordinates. A similar treatment of all of the connections leads to a set of linear constraint equations among the p's which is expressed in matrix form:

$$[A]\{p\} = \{0\} \tag{8}$$

where $[A]$ is a rectangular matrix of constant coefficients. The order of $[A]$ is $k \times m$ where k is the number of equations of constraint and m is the number of p's. Since $m > k$, the matrix may be partitioned in the following way:

$$[A] = [A_1 \vdots A_2]$$

$[A]_1$ is a square matrix of order $k \times k$. Equation (8) is written as follows:

$$[A]_1\{p\}_d + [A_2]\{p\}_f = \{0\} \tag{9}$$

In this equation $\{p\}_f$ represents a subset of $\{p\}$ which is chosen to include independent variables, and $\{p\}_d$ represents a subset of dependent variables. These subsets must be chosen so that the matrix $[A]_1$ is nonsingular. By inverting that matrix, the dependent set may be written in terms of the independent set:

$$\{p\}_d = -[A]_1^{-1}[A]_2\{p\}_f \tag{10}$$

From this, an equation can be derived that relates the complete vector $\{p\}$ to the independent subset.

$$\{p\} = \begin{Bmatrix} p_f \\ p_d \end{Bmatrix} = \left[\begin{array}{c} [I] \\ \hline -[A]_1^{-1}[A]_2 \end{array}\right]\{p\}_f \tag{11}$$

If vector $\{p\}_f$ is identified directly with the system generalized displacement vector $\{q\}$, then the rectangular matrix in Eq. (11) is identified as the transformation matrix $[\beta]$. In general, however, Eq. (11) is considered to accomplish one step in the total transformation and is written as follows:

$$\{p\} = [\beta]'\{p\}_f \tag{12}$$

In many problems it is useful to relate the system generalized coordinates to physically meaningful modal configurations. For example, in structures possessing symmetry it is convenient to define generalized coordinates in terms of symmetrical and antisymmetrical modes. Therefore, a further transformation may be constructed to relate the vector $\{p\}_f$ to the generalized displacement vector $\{q\}$.

$$\{p\}_f = [\beta]''\{q\} \tag{13}$$

The order of both vectors is n. Equations (12) and (13) are combined to yield the complete transformation given by Eq. (7), where

$$[\beta] = [\beta]'[\beta]'' \tag{14}$$

Returning to Eq. (4), the next step in the analysis involves application of the transformation just derived. Substitution of Eq. (7) into Eq. (4), together with corresponding transformations among the velocities and accelerations, followed by premultiplication of all terms by the transposed matrix $[\beta]^T$, yields the following equation:

$$[\beta]^T[m][\beta]\{\ddot{q}\} + [\beta]^T[c][\beta]\{\dot{q}\} + [\beta]^T[k][\beta]\{q\} = [\beta]^T\{P(t)\} \tag{15}$$

This is identified with Eq. (2) and shows how the system properties are synthesized from the properties of the components through application of the transformation matrix $[\beta]$. The following identities are noted:

$$[M] = [\beta]^T[m][\beta] \tag{16}$$

$$[K] = [\beta]^T[k][\beta] \tag{17}$$

$$[C] = [\beta]^T[c][\beta] \tag{18}$$

Also, the system forces are obtained from the component forces by

$$\{Q(t)\} = [\beta]^T\{P(t)\} \tag{19}$$

Component Displacement Functions

In order to formulate the equations of motion for the separate components [see Eq. (3)], it is necessary to establish for each one a system of generalized coordinates. Figure 1 shows a structural component that may be considered typical. It is a "plate-like" structure for which the various displacement modes can be shown clearly. This figure will be used to explain and illustrate the kinematics of a general displacement of an arbitrarily selected point such as P. Figure 1a shows the undisplaced, undeflected structure with a set of constraints indicated by arrows. The constraint system is statically indeterminate with the three constraints i, j, and k considered as redundant. The six numbered constraints 1–6 are selected as a statically determinate set. All of these are movable constraints to which the structure is subjected because of its attachment to other components of the system. The solid outline in Fig. 1b shows the structure after it has been given a rigid-body displacement, which can be considered as having resulted from arbitrary displacements of each of the six statically determinate constraints. In this displacement, point P undergoes the rigid-body vector displacement $\bar{u}^R$. Also, the redundant constraints are displaced as required to maintain the rigid-body configuration. It is clear that, in general, six independent rigid-body displacement modes exist. These may correspond to six independent displacements of the determinate constraints or they may be defined by three translations and three rotations with respect to a set of fixed orthogonal coordinate axes. The structural component may be totally or partially constrained externally so that fewer than six rigid-body modes may exist.

Added to the rigid-body displacements is a set of displacements produced by giving each redundant constraint in turn an arbitrary displacement while keeping all other constraints fixed. The modes produced in this way are called "constraint" modes, and clearly there are as many of them as there are redundant constraints. The dashed outline in Fig. 1b shows the ith constraint mode in which the ith redundant constraint is given a unit displacement while all other constraints remain fixed in the positions attained after the rigid-body displacement. This constraint mode defines a deflected surface in which point P takes on an added increment of deflection $\bar{u}_i^C$. The letter C denotes a "constraint" mode, and the subscript i identifies the ith constraint.

The rigid-body and constraint modes permit the arbitrary displacement of all movable constraints. Beyond these displacements it is necessary to provide for the displacement of other points on the structure relative to the constraints. Such relative displacements are provided by the introduction of a set of independent modes in which all constraints are fixed. An infinity of these modes exists, and it is convenient, although not necessary, to think of these as the "fixed-constraint" natural modes of vibration of the structure. Accordingly, these are called natural modes or normal modes and one of them is shown in Fig. 1c. The corresponding displacement of point P is called $\bar{u}^N$.

The vector displacement of any point $P(x, y, z)$ is found by superposing the three foregoing displacements:

$$\bar{u}(x, y, z) = \bar{u}^R(x, y, z) + \bar{u}^C(x, y, z) + \bar{u}^N(x, y, z) \tag{20}$$

Each of these displacements is defined in terms of a set of normalized displacement functions or modes and a set of generalized displacements; thus,

$$\left.\begin{aligned} \bar{u}^R(x, y, z) &= \sum \bar{\phi}_j{}^R(x, y, z)p_j{}^R \\ \bar{u}^C(x, y, z) &= \sum \bar{\phi}_j{}^C(x, y, z)p_j{}^C \\ \bar{u}^N(x, y, z) &= \sum \bar{\phi}_j{}^N(x, y, z)p_j{}^N \end{aligned}\right\} \tag{21}$$

where

$\bar{\phi}_j{}^R$ = jth rigid-body mode
$\bar{\phi}_j{}^C$ = jth constraint mode
$\bar{\phi}_j{}^N$ = jth natural mode

The p's are generalized displacements expressing the magnitudes of the functions. The modes $\bar{\phi}$ are vector functions of the space coordinates (x, y, z), and each defines a vector displacement field throughout the structure.

In Eqs. (20) and (21) the structure is treated as a continuum in which the functions $\bar{\phi}(x, y, z)$ are continuous functions of the space coordinates. In many practical analyses the structure is discretized so that displacements are defined at only a set of points. In this case the displacement at each point can be written as a component of a column matrix, and Eq. (20) would appear as follows:

$$\{\bar{u}\} = \{\bar{u}^R\} + \{\bar{u}^C\} + \{\bar{u}^N\} \tag{22}$$

The discretized mode functions are displayed in the form of modal matrices $[\bar{\phi}]$ in which the element $\bar{\phi}_{ij}$ is the displacement at point i in the jth mode. Equations (21) take the following matrix form:

$$\left.\begin{aligned} \{\bar{u}^R\} &= [\bar{\phi}^R]\{p^R\} \\ \{\bar{u}^C\} &= [\bar{\phi}^C]\{p^C\} \\ \{\bar{u}^N\} &= [\bar{\phi}^N]\{p^N\} \end{aligned}\right\} \tag{23}$$

If these equations are substituted into Eq. (22), the total displacement column may be written as

$$\{\bar{u}\} = [\bar{\phi}]\{p\} \tag{24}$$

where the complete model matrix appears in partitioned form.

$$[\bar{\phi}] = [\bar{\phi}^R] \mid [\bar{\phi}^C] \mid [\bar{\phi}^N] \tag{25}$$

The column matrix $\{p\}$ is partitioned as follows:

$$\{p\} = \begin{Bmatrix} \{p^R\} \\ \cdots \\ \{p^C\} \\ \cdots \\ \{p^N\} \end{Bmatrix} \tag{26}$$

Using the modes described in the foregoing discussion as generalized coordinates, the generalized mass, stiffness, and damping properties of each component are determined.

Properties of the Components

To determine the generalized properties of the components, the kinetic and strain energies are expressed in generalized coordinates as is the energy dissipated through damping. These energy expressions are introduced into the Lagrange equations to formulate the component equations and to construct the generalized mass, stiffness, and damping matrices for the components. The rth Lagrange equation of the set is written as follows:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_r}\right) - \frac{\partial T}{\partial p_r} + \frac{\partial U}{\partial p_r} + \frac{\partial R}{\mathrm{P}\dot{p}_r} = P_r \tag{27}$$

where

T = kinetic energy
U = strain energy
R = dissipation function for viscous damping.

Mass Matrix

The mass matrix is derived from the kinetic energy terms in Eq. (27). For a continuum, the kinetic energy may be expressed by the following integral, which may be a volume, a surface, or a line integral depending upon the configuration of the structure and the space coordinate system used:

$$T = \frac{1}{2}\int \mu \dot{u}^2 dV \tag{28}$$

where μ = mass density. From Eqs. (20) and (21) the displacement $\bar{u}$ can be written as

$$\bar{u} = \sum_j \bar{\phi}_j p_j \tag{29}$$

Differentiating this to obtain velocities and substituting into Eq. (28) yields

$$T = \frac{1}{2}\sum_j\sum_k \dot{p}_j\dot{p}_k \int \mu\bar{\phi}_j \cdot \bar{\phi}_k dV \tag{30}$$

Since, for small displacements, the kinetic energy is a function of coordinate velocities only, the foregoing quadratic form is expected. The generalized component mass m_{jk} is recognized to have the form

$$m_{jk} = \int \mu\bar{\phi}_j \cdot \bar{\phi}_k dV \tag{31}$$

For a lumped mass model of the structure, the kinetic energy is obtained by summing over the masses.

For this mass representation, the generalized mass matrix for the component is

$$[m] = [\bar{\phi}]^T \cdot \lceil m \rfloor [\bar{\phi}] \tag{32}$$

where $\lceil m \rfloor$ is a diagonal matrix of the masses.

As a consequence of classifying the modes in three categories, namely, rigid-body modes, constraint modes, and normal modes, the mass matrix is partitioned as follows:

$$[m] = \left[\begin{array}{c|c|c} m^{RR} & m^{RC} & m^{RN} \\ \hline m^{CR} & m^{CC} & m^{CN} \\ \hline m^{NR} & m^{NC} & m^{NN} \end{array}\right] \tag{33}$$

To show a single example of the construction of a submatrix, consider the submatrix $[m]^{RC}$. From Eq. (31) the element m_{jk} in this submatrix is given by

$$m_{jk}{}^{RC} = \int \mu \bar{\phi}_j{}^R \cdot \bar{\phi}_k{}^C dV \tag{34}$$

From Eq. (32) the submatrix is given in the following form:

$$[m]^{RC} = [\bar{\phi}^R]^T \cdot [m](\bar{\phi}^C] \tag{35}$$

This results from partitioning the modal matrix as shown in Eq. (25).

Stiffness Matrix

The stiffness matrix is derived from the strain energy term in Eq. (27). The strain energy of a component may be written in terms of the generalized displacements and the generalized stiffness matrix.

$$U = \frac{1}{2}\{p\}^T[k]\{p\} = \frac{1}{2}\sum_i\sum_j k_{ij}p_ip_j \tag{36}$$

It may be expressed also in terms of the stress and strain distributions over the component

$$U = \frac{1}{2}\int \{\tau\}^T\{e\}dV \tag{37}$$

where

$\{\tau\}$ = stress vector containing the six components of stress at a point

$\{e\}$ = strain vector of six components of strain at a point

For materials that follow Hooke's law over the range of stress and strain considered in the analysis, the linear relationship between stress and strain may be written as

$$\{\tau\} = [G]\{e\} \tag{38}$$

where $[G]$ is a square symmetric matrix of coefficients that are dependent upon the elastic properties of the material. When this is inserted into Eq. (37), the strain energy integral has the form

$$U = \frac{1}{2}\int \{e\}^T[G]\{e\}dV \tag{39}$$

Considering the set of normalized modes that comprise the generalized coordinates for the component, it can be seen that each mode is characterized by its own strain distribution. If the strain vector related to the ith mode is denoted by $\{\bar{e}\}_i$, the total strain is found by superposition:

$$\{e\} = \sum_i \{\bar{e}\}_i p_i \tag{40}$$

When this representation is used in Eq. (39) the strain energy integral becomes

$$U = \frac{1}{2}\sum_i\sum_j p_ip_j \int \{\bar{e}\}_i{}^T[G]\{\bar{e}\}_j dV \tag{41}$$

Comparing with Eq. (36) it is seen that the stiffness coefficient k_{ij} may be written as

$$k_{ij} = \int \{\bar{e}\}_i{}^T[G]\{\bar{e}\}_j dV \tag{42}$$

Hooke's law, Eq. (38), also relates the individual modal stresses and strains so that the foregoing equation also may be written as

$$k_{ij} = \int \{\bar{\tau}\}_i{}^T\{\bar{e}\}_j dV \tag{43}$$

This equation may be stated as follows: The stiffness coefficient k_{ij} is equal to the work done by stresses associated with the ith mode acting through strains associated with the jth mode. This is also equal to the work done by the set of external, generalized forces associated with the ith mode acting through displacements associated with the jth mode. This statement will be useful in deducing certain characteristics of the stiffness matrix.

The stiffness matrix may be partitioned in the same way as the mass matrix:

$$[k] = \begin{bmatrix} k^{RR} & k^{RC} & k^{RN} \\ k^{CR} & k^{CC} & k^{CN} \\ k^{NR} & k^{NC} & k^{NN} \end{bmatrix} \tag{44}$$

The submatrices in the first row and first column, involving rigid-body modes, are null matrices. This follows from the foregoing work statement, since the work done by a self-equilibrating force system on a rigid-body displacement is zero. The order $[k]^{RR}$ is equal to the number of rigid-body modes.

The submatrix $[k]^{CC}$ is the stiffness matrix associated with the redundant constraint system because the forces associated with a constraint mode are just the forces at the constraints, and they do work only on displacements of the redundant constraints. Thus, the order of this square matrix is equal to the number of redundant constraints.

The submatrix $[k]^{CN}$ is a null matrix. This is true because the work done by constraint forces on a normal mode displacement is zero because of the fact that in a normal mode the constraints are fixed. Because of symmetry, submatrix $[k]^{NC}$ is also a null matrix.

The submatrix $[k]^{NN}$ is a diagonal matrix if the "normal" modes are the natural modes of vibration. This is true in consequence of the orthogonality of these natural modes. The order of this matrix is arbitrary and depends only upon the number of these modes chosen for the analysis. This, in turn, depends upon accuracy requirements. The ith element of this diagonal matrix is related to the corresponding element of the mass submatrix $[m]^{NN}$, which is also diagonal, by the relationship

$$k_{ii}{}^{NN} = \omega_i{}^2 m_{ii}{}^{NN} \tag{45}$$

where ω_i is the natural frequency of the component in the ith mode obtained with all constraints fixed.

If the so-called "normal" modes are not natural vibration modes, then the matrices $[k]^{NN}$ and $[m]^{NN}$ are not diagonal, and the foregoing relationship between them does not hold. However, as long as these modes satisfy the fixed boundary conditions, the coupling submatrices $[k]^{CN}$ and $[k]^{NC}$ continue to be null matrices.

In consequence of the foregoing results, the partitioned stiffness matrix takes on a simple form:

$$[k] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k^{CC} & 0 \\ 0 & 0 & k^{NN} \end{bmatrix} \tag{46}$$

Damping Matrix

The damping matrix for linear viscous damping is derived from the dissipation term in the Lagrange equation. In damped structures the dissipation of energy can be caused by various mechanisms, including material hysteresis, friction in connections, friction caused by motion through fluids, etc. Such damping is unavoidable, and the mechanisms involved are not generally well understood. The concept of viscous damping is sometimes adopted for the purpose of simplification. For lightly damped systems, this may be regarded as a reasonable choice.

In contrast with unavoidable damping, devices are sometimes employed deliberately to introduce damping for the purpose of reducing dynamic responses. If these devices produce velocity-dependent damping forces, a dissipation function can be derived that depends upon the properties of the dampers and the velocities. The dissipation function may be written also in terms of a generalized damping matrix and the generalized component velocities:

$$R = \tfrac{1}{2}\{\dot{p}\}^T[c]\{\dot{p}\} \tag{47}$$

From this equation an element c_{jk} of the damping matrix may be calculated by differentiation:

$$c_{jk} = \partial^2 R/\partial\dot{p}_j\partial\dot{p}_k \tag{48}$$

If R is known as a function of the physical properties of the system, the foregoing calculation can be carried out. Discussion of the details is not included here inasmuch as the subject of damping mechanisms is not considered to be part of this paper.

The generalized damping matrix is symmetric as are the mass and stiffness matrices. This can be seen from Eq. (48) by observing that the results of the calculation are not changed by interchanging the indices j and k.

It is clear that the damping matrix may be partitioned in the same way as are the mass and stiffness matrices.

$$[c] = \left[\begin{array}{c:c:c} c^{RR} & c^{RC} & c^{RN} \\ \hdashline c^{CR} & c^{CC} & c^{CN} \\ \hdashline c^{NR} & c^{NC} & c^{NN} \end{array}\right] \tag{49}$$

In general, all of the submatrices exist as in the case of the mass matrix. It may be noted, however, that if all the damping forces in the component are internal, i.e., they form a self-equilibrating system of forces within the component, then rigid-body motions are not damped. In this case the submatrices in the first row and the first column will be null matrices.

Forces on the Component

The forces acting on a structural component may be considered in two categories: 1) those imposed through the constraints, and 2) those applied by sources external to the system. The first category includes forces imposed through connections to other components as well as those applied as reactions to the system at fixed constraints. Those applied at connections do work on the displacements of the component, whereas those applied at fixed constraints do not.

Forces imposed through the intercomponent connections are classified in two further categories. Let

R_s^R = force applied through the sth statically determinate constraint
R_r^C = force applied through the rth redundant constraint

Forces applied by sources external to the system may be distributed forces, or they may be point forces. Let $\bar{f}$ represent the intensity of distributed force at any point in the structure and let $\bar{F}_i$ represent a point force at a point whose displacement is $\bar{u}_i$.

The virtual work done by these forces on a virtual displacement of the component is given by

$$\delta W = \sum_s R_s^R\delta\tilde{u}_s^R + \sum_r R_r^C(\delta\tilde{u}_r^R + \delta\tilde{u}_r^C) + \int \bar{f}\cdot\delta\bar{u}dV + \sum_i \bar{F}_i\cdot\delta\bar{u}_i \tag{50}$$

where

$\delta\tilde{u}_s^R$ = virtual displacement of sth statically determinate constraint in a rigid-body mode
$\delta\tilde{u}_r^R$ = virtual displacement of rth redundant constraint in a rigid-body mode
$\delta\tilde{u}_r^C$ = virtual displacement of rth redundant constraint in a constraint mode

Note that displacements of the constraints are distinguished by a tilda above the letter. These displacements are considered as scalar quantities since the constraints themselves are endowed with the property of direction. The constraint displacements may be defined in terms of modal displacements at the constraint:

$$\left.\begin{aligned} \tilde{u}_s^R &= \sum_j \tilde{\phi}_{sj}^{RR}p_j^R \\ \tilde{u}_r^R &= \sum_j \tilde{\phi}_{rj}^{CR}p_j^R \\ \tilde{u}_r^C &= \sum_j \tilde{\phi}_{rj}^{CC}p_j^C \end{aligned}\right\} \tag{51}$$

where

$\tilde{\phi}_{sj}^{RR}$ = modal displacement of sth statically determinate constraint in jth rigid-body mode
$\tilde{\phi}_{rj}^{CR}$ = modal displacement of rth redundant constraint in jth rigid-body mode
$\tilde{\phi}_{rj}^{CC}$ = modal displacement of rth redundant constraint in jth constraint mode

By substituting the expressions (51) into Eq. (50) the following virtual work equation is obtained:

$$\begin{aligned} \delta W = \sum_j \delta p_j^R \Big[\sum_s R_s^R\tilde{\phi}_{sj}^{RR} + \sum_r R_r^C\tilde{\phi}_{rj}^{CR} + \\ \int \bar{f}\cdot\bar{\phi}_j^R dV + \sum_i \bar{F}_i\cdot\bar{\phi}_{ij}^R\Big] + \sum_j \delta p_j^C \times \\ \Big[\sum_r R_r^C\tilde{\phi}_{rj}^{CC} + \int \bar{f}\cdot\bar{\phi}_j^C dV + \sum_i \bar{F}_i\cdot\bar{\phi}_{ij}^C\Big] + \sum_j \delta p_j^N \times \\ \Big[\int \bar{f}\cdot\bar{\phi}_j^N dV + \sum_i \bar{F}_i\cdot\bar{\phi}_{ij}^N\Big] \end{aligned} \tag{52}$$

The virtual work can be written also in terms of generalized forces and displacements:

$$\delta W = \sum_j P_j^R\delta p_j^R + \sum_j P_j^C\delta p_j^C + \sum_j P_j^N\delta p_j^N \tag{53}$$

Comparing Eqs. (52) and (53), with the recognition that the generalized virtual displacements $\delta p_j (j = 1, 2, 3, \ldots)$ are all independent, leads to the following equations for generalized forces:

$$\left.\begin{aligned} P_j^R &= \sum_s R_s^R\tilde{\phi}_{sj}^{RR} + \sum_r R_r^C\tilde{\phi}_{rj}^{CR} + \\ &\qquad \int \bar{f}\cdot\bar{\phi}_j^R dV + \sum_i \bar{F}_i\cdot\bar{\phi}_{ij}^R \\ P_j^C &= \sum_r R_r^C\tilde{\phi}_{rj}^{CC} + \int \bar{f}\cdot\bar{\phi}_j^C dV + \sum_i \bar{F}_i\cdot\bar{\phi}_{ij}^C \\ P_j^N &= \int \bar{f}\cdot\bar{\phi}_j^N dV + \sum_i \bar{F}_i\cdot\bar{\phi}_{ij}^N \end{aligned}\right\} \tag{54}$$

It is convenient to define, separately, a set of generalized external forces as follows:

$$\left.\begin{aligned} \mathfrak{F}_j^R &= \int \bar{f}\cdot\bar{\phi}_j^R dV + \sum_i \bar{F}_i\cdot\bar{\phi}_{ij}^R \\ \mathfrak{F}_j^C &= \int \bar{f}\cdot\bar{\phi}_j^C dV + \sum_i \bar{F}_i\cdot\bar{\phi}_{ij}^C \\ \mathfrak{F}_j^N &= \int \bar{f}\cdot\bar{\phi}_j^N dV + \sum_i \bar{F}_i\cdot\bar{\phi}_{ij}^N \end{aligned}\right\} \tag{55}$$

Equations (54) are written in the following matrix form, using the generalized external forces given by Eqs. (55):

$$\{P\} = [\tilde{\phi}]^T\{R\} + \{\mathfrak{F}\} \tag{56}$$

where

$$\{P\} = \left\{\begin{array}{c} P^R \\ \hdashline P^C \\ \hdashline P^N \end{array}\right\} \qquad \{R\} = \left\{\begin{array}{c} R^R \\ \hdashline R^C \end{array}\right\} \qquad \{\mathfrak{F}\} = \left\{\begin{array}{c} \mathfrak{F}^R \\ \hdashline \mathfrak{F}^C \\ \hdashline \mathfrak{F}^N \end{array}\right\}$$

$$[\tilde{\phi}] = \left[\begin{array}{c:c:c} \tilde{\phi}^{RR} & 0 & 0 \\ \hdashline \tilde{\phi}^{CR} & \tilde{\phi}^{CC} & 0 \end{array}\right]$$

At this point, it is noted, parenthetically, that the constraint displacements are related to the generalized displacements through the matrix $[\tilde{\phi}]$, as follows:

$$\begin{Bmatrix} \tilde{u}_R \\ \tilde{u}_C \end{Bmatrix} = \begin{bmatrix} \tilde{\phi}^{RR} & 0 & 0 \\ \tilde{\phi}^{CR} & \tilde{\phi}^{CC} & 0 \end{bmatrix} \begin{Bmatrix} p^R \\ p^C \\ p^N \end{Bmatrix} \tag{57}$$

where

$\tilde{u}_R$ = the total displacement of a statically determinate constraint
$\tilde{u}_C$ = the total displacement of a redundant constraint

Component Equations

Having developed the component properties and forces in previous sections of the paper, the equations of motion may now be written. Equation (3) has been written for the sth component with viscous damping. This equation is written in partitioned form as follows, using Eqs. (33, 46, 49, and 56):

$$\begin{bmatrix} m^{RR} & m^{RC} & m^{RN} \\ m^{CR} & m^{CC} & m^{CN} \\ m^{NR} & m^{NC} & m^{NN} \end{bmatrix}_s \begin{Bmatrix} \ddot{p}^R \\ \ddot{p}^C \\ \ddot{p}^N \end{Bmatrix}_s + \begin{bmatrix} c^{RR} & c^{RC} & c^{RN} \\ c^{CR} & c^{CC} & c^{CN} \\ c^{NR} & c^{NC} & c^{NN} \end{bmatrix}_s \begin{Bmatrix} \dot{p}^R \\ \dot{p}^C \\ \dot{p}^N \end{Bmatrix}_s +$$
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & k^{CC} & 0 \\ 0 & 0 & k^{NN} \end{bmatrix}_s \begin{Bmatrix} p^R \\ p^C \\ p^N \end{Bmatrix}_s = \begin{bmatrix} \tilde{\phi}^{RR^T} & \tilde{\phi}^{CR^T} \\ 0 & \tilde{\phi}^{CC^T} \\ 0 & 0 \end{bmatrix}_s \begin{Bmatrix} R^R \\ R^C \end{Bmatrix}_s + \begin{Bmatrix} \mathfrak{F}^R \\ \mathfrak{F}^C \\ \mathfrak{F}^N \end{Bmatrix}_s \tag{58}$$

In compact form this equation appears as

$$[m]_s\{\ddot{p}\}_s + [c]_s\{\dot{p}\}_s + [k]_s\{p\}_s = [\tilde{\phi}]_s{}^T\{R\}_s + \{\mathfrak{F}\}_s \tag{59}$$

The set of these equations, written for all of the components, is put together as shown in Eq. (4) and is then operated on by the transformation matrix defined in Eq. (7) to form the equation of motion for the system. This operation is shown in Eq. (15). The system equation is solved to yield the time-dependent system generalized response vectors $\{q\}$, $\{\dot{q}\}$, and $\{\ddot{q}\}$. These are introduced into the transformation equation (7) to yield the corresponding time-dependent component response vectors $\{p\}$, $\{\dot{p}\}$, and $\{\ddot{p}\}$. With these vectors available, the connection forces may be found using Eq. (58). It is most convenient to solve first for the forces at the redundant constraints by using the second of the three matrix equations represented in Eq. (58).

$$\{R^C\}_s = [\tilde{\phi}^{CC}]_s{}^{T-1}([m^{CR}]_s\{\ddot{p}^R\}_s + [m^{CC}]_s\{\ddot{p}^C\}_s + [m^{CN}]_s\{\ddot{p}^N\}_s + [c^{CR}]_s\{\dot{p}^R\}_s + [c^{CC}]_s\{\dot{p}^C\}_s + [c^{CN}]_s\{\dot{p}^N\}_s + [k^{CC}]_s\{p^C\}_s - \{\mathfrak{F}^C\}_s) \tag{60}$$

Following this calculation the result may be introduced into the first of the three equations in (58) to solve for the forces at the statically determinate constraints.

$$\{R^R\}_s = [\tilde{\phi}^{RR}]^{T-1}([m^{RR}]_s\{\ddot{p}^R\}_s + [m^{RC}]_s\{\ddot{p}^C\}_s + [m^{RN}]_s\{\ddot{p}^N\}_s + [c^{RR}]_s\{\dot{p}^R\}_s + [c^{RC}]_s\{\dot{p}^C\}_s + [c^{RN}]_s\{\dot{p}^N\}_s - [\tilde{\phi}^{CR}]_s{}^T\{R^C\}_s - \{\mathfrak{F}^R\}_s) \tag{61}$$

Under the time-dependent constraint forces and the external forces, each component is in a condition of dynamic equilibrium. Reactions at fixed constraints that may exist are not found by the foregoing calculations because these do no work on component displacements; hence they do not enter into the virtual work expressions. They can be determined, however, by writing equilibrium equations for the component or components to which they apply.

Equilibrium of the Connection Forces

It can be shown that, through the use of the condition of continuity at the component connections, the equilibrium of the interacting forces at these connections is implicitly assured. To show that this is so, the steps leading to the determination of the system generalized forces must be examined. Substitution of Eq. (56) into Eq. (19) shows that the force vector for the system is

$$\{Q(t)\} = [\beta]^T[\tilde{\phi}]^T\{R\} + [\beta]^T\{\mathfrak{F}\} \tag{62}$$

In this equation, the matrix $[\tilde{\phi}]$ is composed of the component matrices arranged as shown:

$$[\tilde{\phi}] = \begin{bmatrix} [\tilde{\phi}]_1 & & & & & & \\ & [\tilde{\phi}]_2 & & & & & \\ & & \ddots & & & & \\ & & & [\tilde{\phi}]_r & & & \\ & & & & [\tilde{\phi}]_s & & \\ & & & & & \ddots & \end{bmatrix} \tag{63}$$

Similarly, the vectors $\{R\}$ and $\{\mathfrak{F}\}$ include those for all of the separate components arranged as follows:

$$\{R\} = \begin{Bmatrix} \{R\}_1 \\ \{R\}_2 \\ \vdots \\ \{R\}_r \\ \{R\}_s \\ \vdots \end{Bmatrix} \quad \text{and} \quad \{\mathfrak{F}\} = \begin{Bmatrix} \{\mathfrak{F}\}_1 \\ \{\mathfrak{F}\}_2 \\ \vdots \\ \{\mathfrak{F}\}_r \\ \{\mathfrak{F}\}_s \\ \vdots \end{Bmatrix} \tag{64}$$

Since the components of the system vector $\{R\}$ are forces in the component connections, which are internal forces within the system, they should not contribute to the generalized system force $\{Q(t)\}$. Therefore, the first term on the right side of Eq. (62) should vanish, and the system force should be simply

$$\{Q(t)\} = [\beta]^T\{\mathfrak{F}\} \tag{65}$$

The preceding statement concerning the self-equilibrating property of internal forces will be accepted generally without proof; hence a rigorous proof is not included here. Such a proof may be constructed, however, by showing that the first term on the right side of Eq. (62) does indeed vanish, or

$$[\beta]^T[\tilde{\phi}]^T\{R\} = \{0\} \tag{66}$$

The proof may proceed by applying the principle of virtual work to two mutually connected components of the system. It is almost trivial to show that the two reactions at a common connection are equal in magnitude and oppositely directed, provided the condition of displacement continuity is assured. From this point it is easy to show that Eq. (66) must hold.

This matter is of importance in dealing with the present method of analysis only in pointing out that the explicit use of force equilibrium equations is not necessary in deriving the equations of constraint embodied in Eq. (8).

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Liquid Surface Oscillations in Longitudinally Excited Rigid Cylindrical Containers

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The results of a theoretical and experimental study of low-frequency liquid surface motions in a longitudinally vibrated tank are presented and discussed in some detail. Large-amplitude free surface motions occur when the liquid responds as a one-half subharmonic of the excitation. This form of response exhibits "jump" phenomena common to nonlinear systems. Harmonic and superharmonic liquid surface motions are also observed, although their amplitudes are usually smaller than those corresponding to the one-half subharmonic response. Comparison between theoretical and experimental results are given, in most instances the correlation being rather close.

Nomenclature§

$\bar{a}_{mn}(a_{mn})$	=	amplitude of the m, nth liquid surface mode
$\bar{\alpha}_{mn}(\alpha_{mn})$	=	amplitude of the m, nth component of the liquid velocity potential
d]	=	tank diameter
$h(b)$	=	liquid depth
J_m	=	mth order Bessel function of first kind
$\bar{\lambda}_{mn}(\lambda_{mn})$	=	defined by $J_m'(\lambda_{mn}R) = 0$
$N\omega$	=	tank excitation frequency (N is a positive number)
$\bar{\phi}(\phi)$	=	liquid velocity potential
$\bar{r}$, θ, $\bar{z}(r, \theta, z)$	=	tank fixed coordinate system
R	=	tank radius
σ	=	frequency parameter, ω/ω_{kl}
$\bar{t}(t)$	=	time
$\bar{u}$, $\bar{v}$, $\bar{w}$	=	fluid velocity components
$X_0(\epsilon)$	=	tank excitation amplitude
$\bar{y}(y)$	=	free surface displacement above mean level of liquid
Y_0	=	one-half the sum of the maximum liquid surface excursion in the positive and negative direction

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§ Symbols in parenthesis are the nondimensional equivalents of the preceding quantity.

I. Introduction

THE importance of liquid sloshing on the over-all dynamics of liquid fuel rockets is well recognized, and extensive investigations of this problem have been conducted and reported. Papers given by Abramson[1] and Cooper[2] review much of this work.

Liquid sloshing can be excited by a variety of motions of the liquid container; of these, longitudinal oscillations of the vehicle and the subsequent liquid response generally have been the least investigated. This is probably a result of the fact that linear analysis, which adequately predicts resonant frequencies and liquid responses for most types of container motion, fails to predict the liquid response amplitudes for longitudinal forced motion. Some previous linear analyses and experiments, however, have yielded results that give the over-all liquid behavior, with the exception of response amplitude, and vividly describe the complexity of this problem.

A nonlinear analysis, which predicts the complete liquid response, has been performed for an infinitely deep, very narrow rectangular tank by Yarymovych and Skalak.[3] No such nonlinear analysis has been conducted previously for the circular cylindrical tank. The purpose of the present paper, therefore, is to present a comprehensive nonlinear analytical and experimental study of liquid sloshing in a longitudinally excited, finite depth, rigid, circular cylindrical tank with emphasis placed chiefly on low-frequency excitation and the corresponding liquid response.

II. Interpretation of Mathieu Stability Chart

The frequency of the liquid surface motion for most forms of forced vibration of a container usually corresponds to the excitation frequency, and the amplitude of the motion can be calculated accurately by a linear analysis. A nonlinear analysis is necessary here only when large amplitude motions occur in the vicinity of a free surface resonance. In con-